$\alpha > 0$ and the use of the Weierstrass inequality show that the discontinuity in the function $u_1(x, y)$ holds only for $x = x^*$ in this case, and this function has the form

$$u_1(x, y) = \begin{cases} F, & x > x^* \\ -F, & x < x^* \end{cases}$$

In case $\varepsilon < wx^*$ the functional I equals the following:

$$I = \frac{2Fl^2\alpha}{\pi^2 w}\left(2 - \sin\frac{\pi wT}{l}\right) + FT - \frac{F\varepsilon}{w} \tag{4.9}$$

Upon compliance with the inequality $-{}^1/_4\, \varepsilon^{-1} < \alpha < 0$ the distribution of the values of the function $u_1(x, y)$ is as shown in Fig. 7. The functional I hence has the value

$$I = \frac{4Fl^2\alpha}{\pi w}\left(\frac{wT}{l} - \frac{\varepsilon}{2l} - \frac{1}{\pi} + \frac{1}{2\pi}\sin\frac{\pi wT}{l} - \frac{1}{2\pi}\sin\frac{2\pi\varepsilon}{l}\right) + FT - \frac{F\varepsilon}{w} \tag{4.10}$$

Comparing (4.9) and (4.10) for $\alpha = 0$, we obtain the following: $I = FT - F\varepsilon / w$. Let us note that again the problem has an innumerable set of solutions for $\alpha = 0$, two of which are described above.

BIBLIOGRAPHY

1. Petukhov, L. V. and Troitskii, V. A., Variational optimization problems for equations of hyperbolic type. PMM Vol. 36, №4, 1972.
2. Bliss, G. A., Lectures on Calculus of Variations. Moscow, IIL, 1950.
3. Courant, R. and Hilbert, D. L. Methods of Mathematical Physics, Vol. 2. Moscow-Leningrad, Gostekhizdat, 1945.

Translated by M. D. F.

UDC 539.3

PRESSURE OF A STAMP ON A HALF-PLANE WITH INCLUSIONS

PMM Vol. 36, №5, 1972, pp. 905-912
Iu. A. AMENZADE
(Baku)
(Received June 8, 1971)

The problem of pressing a stamp on a half-plane with holes in which inclusions from another material are inserted with prestress, is considered. The cases of frictionless contact and for total adhesion of the stamp to the half-plane are examined. It is shown that when the elastic constants of the half-plane and inclusions are identical, the auxiliary functions introduced on the contours are defined completely by the magnitude of the prestress and the solution of the problem is obtained in closed form. If the elastic constants are distinct, then the method proposed results in some functional relationships which can be used to determine the auxiliary functions from the kinematic contact conditions.

1. Formulation of the problem. Let us consider an elastic half-plane S_0 with a finite number of holes. The half-plane is bounded by a line L_0, and the holes by simple smooth curves L_k $(k = 1, 2, \ldots, m)$ without common points nor with the line L_0. Let us also assume that inclusions S_k $(k = 1, 2, \ldots, m)$ of the same form as the holes are inserted with prestress. For simplicity it is assumed that the inclusions S_k are simply-connected. The half-plane is subjected to a rigid pressed-in stamp. Let us consider the line outside the stamp to be forcefree, the base of the stamp to be flat, and the stamp to be able to move translationally. It is hence assumed that the force acting on the stamp is sufficiently large so as to assure that the whole stamp will touch the boundary L_0 of the half-plane.

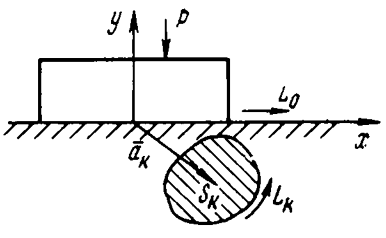

Fig. 1

Let $2a$ be the width of the stamp, P the force with which the stamp is impressed into the half-plane, $\bar{a}_k$ the affix of the center of the hole, $\bar{a}_k = d_k - ih_k$. Furthermore, let $\varkappa_0$, μ_0 and $\varkappa_k$, μ_k denote the elastic constants of the materials filling the domains S_0 and S_k (Fig. 1). A method of solution is given below which permits reducing this problem to a contact problem for a simply-connected half-plane.

2. Frictionless contact problems. According to investigations [1], we have for these problems

on L_0

$$\operatorname{Im}[t\Phi_0'(t) + \Psi_0(t)] = 0, \qquad (y = 0,\ |t| > a) \tag{2.1}$$

$$\operatorname{Re}[2\Phi_0(t) + t\Phi_0'(t) + \Psi_0(t)] = 0 \qquad (y = 0,\ |t| > a) \tag{2.2}$$

$$\operatorname{Im}[\varkappa_0\Phi_0(t) - \overline{\Phi_0(t)} - t\overline{\Phi'_0(t)} - \overline{\Psi_0(t)}] = 0 \qquad (y = 0,\ |t| < a) \tag{2.3}$$

$$\operatorname{Im}[\bar{t}\Phi_0'(t) + \Psi_0(t)] = 0 \qquad (y = 0,\ |t| < a) \tag{2.4}$$

on L_k

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = \varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)} \tag{2.5}$$

$$\varkappa_0\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)} = C_k[\varkappa_k\varphi_k(t) - t\overline{\varphi_k'(t)} - \overline{\psi_k(t)}] + 2\mu_0 g_k(t) \tag{2.6}$$

Here $\Phi_0(z) = \varphi_0'(z)$, $\Psi_0(z) = \psi_0'(z)$, $C_k = \mu_0/\mu_k$, and $g_k(t)$ is a specified function.

Let us introduce a new regular function

$$G_0(z) = z\Phi_0'(z) + \Psi_0(z) \tag{2.7}$$

in the domain S_0. Taking account of (2.7), we obtain from (2.1) and (2.4)

$$\operatorname{Im} G_0(t) = 0 \qquad (y = 0,\ -\infty \leqslant t \leqslant +\infty) \tag{2.8}$$

taking account of (2.4), we determine from (2.3)

$$\operatorname{Im} \Phi_0(t) = 0 \qquad (y = 0,\ -a < t < a) \tag{2.9}$$

and from (2.2) we have

$$\operatorname{Re}[2\Phi_0(t) + G_0(t)] = 0 \qquad (y = 0,\ |t| > a \tag{2.10}$$

Therefore, conditions (2.5) – (2.7), (2.9), (2.10) determine the problem posed.

We introduce the new unknown functions [2]

$$\omega_k(t) = {}^1/_2\,[\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)} - \varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)}] \quad (2.11)$$

on L_k. We then find from (2.5), (2.11)

$$\varphi_0(t) = \varphi_k(t) + \omega_k(t),\quad \psi_0(t) = \psi_k(t) - \overline{\omega_k(t)} - \bar{t}\,\omega_k'(t) \quad (2.12)$$

If the elastic constants of the half-plane and the inclusions are identical ($\varkappa_k = \varkappa_0 = \varkappa$, $\mu_k = \mu_0 = \mu$), then the functions $\omega_k(t)$ are determined completely by the magnitude of the elastic prestress and are considered known. In this case, as is known in the Appendix, the solution of the problem posed is obtained in closed form. If the elastic constants are distinct, then the method presented results in the functional relationship (2.26), which can be used to determine the auxiliary functions $\omega_k(t)$ from (2.6).

In conformity with the properties of Cauchy type integrals and the theorem on analytic continuation, we introduce the following regular functions in the simply connected domain $S_0 + S_1 + \ldots + S_m$:

$$\varphi(z) = \begin{cases}\varphi_0(z) + J_1(z), & z \in S_0\\ \varphi_\nu(z) + J_1(z), & z \in S_\nu\end{cases} \qquad (\nu = 1, 2, \ldots, m) \quad (2.13)$$

$$\psi(z) = \begin{cases}\psi_0(z) + J_2(z), & z \in S_0\\ \psi_\nu(z) + J_2(z), & z \in S_\nu\end{cases} \qquad (\nu = 1, 2, \ldots, m) \quad (2.14)$$

Here

$$J_1(z) = \sum_{k=1}^{m} \frac{1}{2\pi i}\int_{L_k} \frac{\omega_k(t)}{t - z}\,dt$$

$$J_2(z) = -\sum_{k=1}^{m} \frac{1}{2\pi i}\int_{L_k} \frac{\overline{\omega_k(t)} + \bar{t}\,\omega_k'(t)}{t - z}\,dt \quad (2.15)$$

We differentiate (2.13), (2.14) by introducing the notation $\varphi'(z) = \Phi(z)$ and $\psi'(z) = \Psi(z)$, and we convert (2.8) – (2.10) into

$$\operatorname{Im} G(t) = \operatorname{Im}[tJ''(t) + J_2'(t)] \qquad (y = 0,\ |t| \leqslant \infty) \quad (2.16)$$

$$\operatorname{Im}\Phi(t) = \operatorname{Im} J_1'(t) \qquad (y = 0,\ |t| < a) \quad (2.17)$$

$$\operatorname{Re}[2\Phi(t) + G(t)] = \operatorname{Re}[2J_1'(t) + tJ_1''(t) + J_2'(t)] \qquad (y = 0,\ |t| > a) \quad (2.18)$$

Here $G(z) = z\Phi'(z) + \Psi(z)$ is a regular function in the simply-connected domain $S_0 + S_1 + \ldots + S_m$. Let us assume that $\Phi(z) = O(1/z)$ and $\Psi(z) = O(1/z)$ in the neighborhood of $z = \infty$; then $G(z) = O(1/z)$ also. We give condition (2.18) the form

$$G^-(t) - \overline{G^+(t)} = f_1^-(t) - \overline{f_1^+(t)} \qquad (y = 0,\ |t| \leqslant \infty) \quad (2.19)$$

$$f_1(z) = zJ_1''(z) + J_2'(z)$$

On the basis of (2.15), we obtain $f_1(z) = O(1/z^2)$ in the neighborhood of $z = \infty$, hence $G(z) = -\bar{f}_1(z)$. Hence, it follows that the conditions (2.17) and (2.18) can be written in the form

$$\Phi^-(t) - \overline{\Phi^-(t)} = f_1^-(t) - \overline{f_1^-(t)} \qquad (y = 0, |t| < a) \tag{2.20}$$

$$\Phi^-(t) + \overline{\Phi^-(t)} = f_1^-(t) + \overline{f_1^-(t)} + f_2^-(t) + \overline{f_2^-(t)} \qquad (y = 0, |t| > a) \tag{2.21}$$

where $f_2(z) = J_1'(z)$. We take the regular function

$$F(z) = \begin{cases} \Phi(z) + f_2(z), & \operatorname{Im} z < 0 \\ \overline{\Phi(z)} + f_2(z), & \operatorname{Im} z > 0 \end{cases} \tag{2.22}$$

We then have from (2.20) and (2.21)

$$F^+(t) - F^-(t) = 0 \qquad (y = 0, |t| < a) \tag{2.23}$$

$$F^+(t) - bF^-(t) = f(t) \qquad (y = 0, |t| > a) \tag{2.24}$$

Here

$$b = -1, \quad f(t) = 2[f_2^-(t) + \overline{f_2^-(t)}] + f_1^-(t) + \overline{f_1^-(t)} \tag{2.25}$$

We conclude from (2.23) that the function $F(z)$ is regular on the z-plane slit along L $((a, \infty), (-\infty, -a))$. Such a function should be determined from the Riemann problem (2.24) with index $b = -1$. The solution of this problem which equals zero at infinity has the form [1]

$$F(z) = \frac{1}{\sqrt{z^2 - a^2}} \frac{1}{2\pi i} \int_L \frac{f(t)\,[\sqrt{t^2 - a^2}]^{(-)}}{t - z}\, dt + \frac{C_0}{\sqrt{z^2 - a^2}}, \qquad C_0 = -\frac{iP}{2\pi} \tag{2.26}$$

As has already been noted above, (2.26) permits obtaining a final solution of the problem only in the case of identical elastic constants for the half-plane and the inclusions.

3. Contact problems in the case of total adhesion under the stamp. In this case the following conditions hold on L_0 [1]:

$$\begin{aligned} \Phi_0^+(t) - \Phi_0^-(t) &= 0 \qquad (y = 0, |t| > a) \\ \Phi_0^+(t) - \varkappa\Phi_0^-(t) &= 0 \qquad (y = 0, |t| < a) \end{aligned} \tag{3.1}$$

The function $\Phi_0(z)$ is defined in the upper half-plane $\bar{S}_0$ as follows:

$$\Phi_0(z) = -\overline{\Phi}_0(z) - z\overline{\Phi}_0'(z) - \overline{\Psi}_0(z), \quad z \in \bar{S}_0$$

Using the notation introduced earlier, let us give conditions (3.1) the form

$$\begin{aligned} \Phi^+(t) - \Phi^-(t) &= f(t) \qquad (y = 0, |t| > a) \\ \Phi^+(t) + \varkappa\Phi^-(t) &= f(t) \qquad (y = 0, |t| < a) \end{aligned} \tag{3.2}$$

where the function $\Phi(z)$ in the upper half-plane $\bar{S}_0$ and the function $f(t)$ are defined as follows:

$$\Phi(z) = -\overline{\Phi}(z) - z\overline{\Phi}'(z) - \overline{\Psi}(z), \qquad z \in \bar{S}_0$$

$$f(t) = -f_2^-(t) - \overline{f_2^+(t)} - \overline{f_1^-(t)} \qquad (y = 0, |t| > a)$$

$$f(t) = \varkappa f_2^-(t) - \overline{f_2^+(t)} - \overline{f_1^-(t)} \qquad (y = 0, |t| < a)$$

The problem (3.2) is a Riemann problem with discontinuous coefficient. The particular solution of the corresponding homogeneous problem is:

$$X_0(z) = (z + a)^{-\gamma} (z - a)^{\gamma - 1}, \quad \gamma = {}^1/_2 + \ln \varkappa / 2\pi i$$

The branch for which $\lim zX_0(z) = 1$ as $z \to \infty$ is selected for the function

$X_0(z)$. The solution of the inhomogeneous problem (2.2) has the form:

$$\Phi(z) = \frac{X_0(z)}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{X_0^+(t)}\frac{dt}{t-z} + C_0X_0(z) \qquad (3.3)$$

where $\Phi(z)$ is a holomorphic function in the expanded z-plane slit along $(-a, a)$ and equal to zero at infinity, $C_0 = iP/2\pi$. Then under the stamp $(|x| < a)$

$$\sigma_y - i\tau_{xy} = \Phi^-(t) - \Phi^+(t) - f_2^-(t) - \bar{f}_2^+(t) - \bar{f}_1^+(t) \qquad (3.4)$$

As in Sect. 2, formula (3.3) can be applied directly only in the case of identical elastic constants.

4. Appendix. Let domains S_ν in the form of circles of identical radius r for which the affixes of the centers are $\bar{a}_\nu = [\nu - 1/2(m+1)]d - ih$ $(\nu = 1, \ldots, m)$ be inserted in a half-plane with identical prestress δ. Let us also assume that $\mu_0 = \mu_\nu = \mu$, $\varkappa_0 = \varkappa_\nu = \varkappa$, that the stamp can be displaced vertically only, and there is no friction below it. Then

$$2\mu g_\nu(t) = 2K(t - \bar{a}_\nu), \qquad K = \mu\delta / r$$

For this problem

$$\omega_k(t) = \varphi_0(t) - \varphi_k(t) = 2K(1+\varkappa)^{-1}(t - \bar{a}_k)$$

The functions $\varphi(z)$ and $\psi(z)$ defined by (2.13), (2.14) are:

$$\varphi(z) = \begin{cases} \varphi_0(z), & z \in S_0 \\ \varphi_\nu(z) + \dfrac{2K}{1+\varkappa}(z - \bar{a}_\nu), & z \in S_\nu \end{cases}$$

$$\psi(z) = \begin{cases} \psi_0(z) + \dfrac{2K}{1+\varkappa}\displaystyle\sum_{k=1}^{m} \frac{2r^2}{z - \bar{a}_k}, & z \in S_0 \\ \psi_\nu(z) + \dfrac{2K}{1+\varkappa}\left(\displaystyle\sum_{k=1}^{m} \frac{2r^2}{z - \bar{a}_k} - a_\nu\right), \quad k \neq \nu, & z \in S_\nu \end{cases}$$

Therefore, we have in the domain S_0

$$\Phi(z) = \Phi_0(z) + J_1'(z), \qquad \Psi(z) = \Psi_0(z) + J_2'(z) \qquad (4.1)$$

$$J_1'(z) = f_2(z) = 0, \quad J_2'(z) = f_1(z) = -\sum_{k=1}^{n} \frac{4Kr^2}{1+\varkappa}\frac{1}{(z - \bar{a}_k)^2} \qquad (4.2)$$

From (2.25) and (4.2), we find

$$f(t) = -\frac{4Kr^2}{1+\varkappa}\sum_{k=1}^{m}\left(\frac{1}{(t - \bar{a}_k)^2} + \frac{1}{(t - a_k)^2}\right) \qquad (4.3)$$

Let us substitute (4.3) into (2.26), then

$$F(z) = \left(-\frac{4Kr^2}{1+\varkappa}J(z) + C_0\right)\frac{1}{\sqrt{z^2 - a^2}}$$

$$J(z) = \sum_{k=1}^{m} \frac{1}{2\pi i}\int_L \sqrt{t^2 - a^2}\left(\frac{1}{(t - \bar{a}_k)^2} + \frac{1}{(t - a_k)^2}\right)\frac{dt}{t - z}$$

By virtue of the theorem on residues we finally find

$$\Phi_0(z) = -\frac{2Kr^2}{1+\varkappa}\sum_{k=1}^{m}\left\{\frac{1}{(z-\bar{a}_k)^2}+\frac{1}{(z-a_k)^2}+\frac{1}{\sqrt{z^2-a^2}}\times\right.$$
$$\left.\times\left[\frac{a^2-\bar{a}_k z}{\sqrt{\bar{a}_k^2-a^2}\,(\bar{a}_k-z)^2}+\frac{a^2-a_k z}{\sqrt{a_k^2-a^2}\,(a_k-z)^2}\right]\right\}+\frac{C_0}{\sqrt{z^2-a^2}}$$
$$\Psi_0(z) = -z\Phi_0'(z)+\frac{4Kr^2}{1+\varkappa}\sum_{k=1}^{m}\left[\frac{1}{(z-\bar{a}_k)^2}+\frac{1}{(z-a_k)^2}\right]$$

The stress under the stamp $(|x|<a)$ will be

$$\sigma_y = -\frac{2Kr^2}{1+\varkappa}\frac{2}{ai\sqrt{1-(x/a)^2}}\sum_{k=1}^{m}\left[\frac{a^2-\bar{a}_k x}{\sqrt{\bar{a}_k^2-a^2}\,(\bar{a}_k-x)^2}+\right.$$
$$\left.+\frac{a^2-a_k x}{\sqrt{a_k^2-a^2}\,(a_k-x)^2}\right]-\frac{P}{\pi a}\frac{1}{\sqrt{1-(x/a)^2}}$$

Taking into account that $\sqrt{\bar{a}_k^2-a^2} = -\overline{\sqrt{a_k^2-a^2}}$, we obtain

$$\sigma_y = -\frac{8Kr^2}{1+\varkappa}\frac{1}{a\sqrt{1-(x/a)^2}}\sum_{k=1}^{m}\operatorname{Im}\frac{a^2-a_k x}{\sqrt{a_k^2-a^2}\,(a_k-x)^2}-\frac{P}{\pi a}\frac{1}{\sqrt{1-(x/a)^2}} \tag{4.4}$$

Let us consider the case of one inclusion $(a_1 = d - ih)$, then

$$\sigma_y = \left(\frac{8K}{1+\varkappa}\frac{h}{a}\left(\frac{r}{h}\right)^2 B-\frac{P}{\pi a}\right)\frac{1}{\sqrt{1-(x/a)^2}} \tag{4.5}$$

Here

$$B = \frac{1}{\sqrt{\rho}}\left[(\alpha-q\beta)^2-1\right]^{-2}\left(B_1\sin\frac{\varphi}{2}-B_2\cos\frac{\varphi}{2}\right)$$
$$B_1 = (\beta^2-\alpha\beta q)\left[(\alpha-\beta q)^2-1\right]-2(\alpha-\beta q)\beta q$$
$$B_2 = \beta q\left[(\alpha-\beta q)^2-1\right]+2(\alpha-\beta q)(\beta^2-\alpha\beta q)$$
$$\rho = \left[(\alpha^2-\beta^2-1)^2+4\alpha^2\right]^{1/2}, \qquad \operatorname{tg}\varphi = \frac{2\alpha}{\alpha^2-\beta^2-1}$$
$$d/h = \alpha, \quad a/h = \beta, \quad x/a = q$$

To assure translational motion of the stamp, the distance x_c of the line of action of the force P from the stamp axis y is determined from the statics condition

$$x_c = -\frac{1}{P}\int_{-a}^{a}\sigma_y x\,dx \tag{4.6}$$

or

$$x_c = D\int_{-1}^{1}\frac{1}{[(q-q_1)(q-\bar{q}_1)]^2\sqrt{1-q^2}}\Big(\{(\beta^2-\alpha\beta q)\left[(\alpha-\beta q)^2-1\right]-2\beta q\times$$
$$\times(\alpha-\beta q)\}\sin\frac{\varphi}{2}+\{\beta q\left[(\alpha-\beta q)^2-1\right]+2(\alpha-\beta q)(\beta^2-\alpha\beta q)\}\cos\frac{\varphi}{2}\Big)q\,dq$$

where

$$D = -\frac{8}{1+\varkappa}\frac{ka^2}{p}\frac{1}{\beta^5}\left(\frac{r}{h}\right)^2\left[(\alpha-\beta-1)^2+4\alpha^2\right]^{-1/2}$$

The points $q_1 = \alpha/\beta + i\,1/\beta$ and $\bar{q}_1 = \alpha/\beta - i\,1/\beta$ are double poles of the integrand. For sufficiently large $|z| = R$ we have

$$\sqrt{1-z^2} = -i\,\mathrm{R}e^{i\theta}, \qquad \sqrt{1-\bar{z}^2} = i\,\mathrm{R}e^{-i\theta} \tag{4.7}$$

On this basis of the theorem on residues and taking account of (4.5) and (4.7) we obtain from (4.6):

$$x_c = \frac{8a^2K\pi}{(1+\varkappa)\rho}\left(\frac{r}{h}\right)^2\frac{1}{P}\left\{\left(\cos\frac{\varphi}{2}-\alpha\sin\frac{\varphi}{2}\right)+\frac{1}{4\sqrt{\rho}}\left[D_1\sin\varphi+\right.\right.$$

$$\left.+D_2\cos^2\frac{\varphi}{2}-D_3\sin^2\frac{\varphi}{2}\right]-\frac{1}{4\rho^{3/2}}\left[\left(D_4\sin\frac{\varphi}{2}+D_5\cos\frac{\varphi}{2}\right)\times\right.$$

$$\left.\left.\times\left(6\alpha\cos\frac{3\varphi}{2}+2A_7\sin\frac{3\varphi}{2}\right)-\left(D_6\sin\frac{\varphi}{2}+D_7\cos\frac{\varphi}{2}\right)\left(2A_7\cos\frac{3\varphi}{2}-6\alpha\sin\frac{3\varphi}{2}\right)\right]\right\}$$

The constants in this formula are:

$$D_1 = \beta^2A_1 - 2A_2A_9 + 3A_1A_3 - 4A_8$$

$$D_2 = 4\alpha(\beta^2 - A_2 + 2A_4), \qquad D_3 = 4\alpha(A_2 - 3A_3 + 2A_5)$$

$$D_4 = \alpha(\beta^2A_1 - A_1A_2 + A_3A_4 - A_6)$$

$$D_5 = 2\alpha^2\beta^2 - A_1A_2 + A_6, \qquad D_6 = \beta^2A_1 - 2\alpha^2A_2 + A_3A_5 - 4\alpha^2A_1$$

$$D_7 = 2\alpha(\beta^2 + 2A_1 - A_2)$$

$$A_1 = \alpha^2 - 1, \qquad A_2 = 2\beta^2 + \alpha^2 + 1, \qquad A_3 = \beta^2 + 2\alpha^2 + 2$$

$$A_4 = \alpha^2 - 3, \qquad A_5 = 3\alpha^2 - 1, \qquad A_6 = \alpha^4 - 6\alpha^2 + 1$$

$$A_7 = \beta^2 - \alpha^2 + 2, \qquad A_8 = \alpha^4 + 1, \qquad A_9 = \alpha^2 + 1$$

The distance of the line of action of the force P from the y-axis is $x_c = 0.1701a$ for $d/h = 1$, $a/h = 1/3$. The pressure under the stamp has been computed for $P/\pi a = 8K(1+\varkappa)^{-1}(r/h)^2$ for the cases $d = 0$, $a/h = 1/3$; $d/h = 1/3$, $a/h = 1/3$; $d = h$, $a/h = 1/3$ and the diagrams $\sigma_y^* = 1/8\,(1+\varkappa)\,K^{-1}(h/r)^2\sigma_y$ are shown in Fig. 2 by a solid, dashed, and dashed-dot line, respectively. The diagram shown by the heavy solid line is when there is no inclusion with prestress.

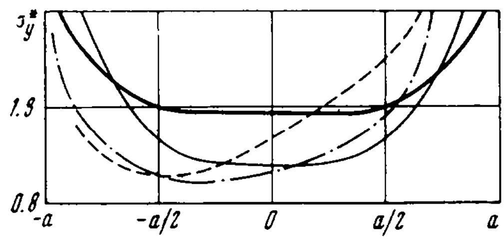

Fig. 2

4.2. Let us solve the problem 1, when total adhesion holds under the stamp. In this case

$$f(t) = -\bar{f}_1(t) = \frac{4Kr^2}{1+\varkappa}\sum_{k=1}^{m}\frac{1}{(z-a_k)^2}$$

$$\Phi(z) = \left(\frac{4Kr^2}{1+\varkappa}J(z) + \frac{i}{2\pi}P\right)X_0(z)$$

$$J(z) = \sum_{k=1}^{m}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dt}{X_0^+(t)(t-a_k)^2(t-z)}$$

On the basis of the residue theorem we have

$$J(z)=\begin{cases}\sum_{k=1}^{m}\left[\frac{d}{dt}\frac{1}{X_0^+(t)(t-z)}\right]_{t=a_k}=J_*(z), & y<0\\ \frac{1}{X_0(z)}\sum_{k=1}^{m}\frac{1}{(z-a_k)^2}+J_*(z), & y>0\end{cases}$$

$$J_*(z)=\sum_{k=1}^{m}\left\{\left[\gamma\left(\frac{a_k+a}{a_k-a}\right)^{\gamma-1}+(1-\gamma)\left(\frac{a_k+a}{a_k-a}\right)^{\gamma}\right]\frac{1}{a_k-z}-\left(\frac{a_k+a}{a_k-a}\right)^{\gamma}\frac{a_k-a}{(a_k-z)^2}\right\}$$

Taking into account that $f_2(t)=0$, on the basis of (2.4) we have under the stamp

$$\sigma_y-i\tau_{xy}=\Phi^-(t)-\Phi^+(t)+\frac{4Kr^2}{1+\varkappa}\sum_{k=1}^{m}\frac{1}{(x-a_k)^2}$$

Hence

$$\sigma_y-i\tau_{xy}=\frac{4Kr^2}{1+\varkappa}\left\{J^-(x)X_0^-(x)-J^+(x)X_0^+(x)+\sum_{k=1}^{m}\frac{1}{(x-a_k)^2}\right\}+\frac{i}{2\pi}P\{X_0^-(x)-X_0^+(x)\}$$

Taking into account that $X^+(x)=-\varkappa X^-(x)$ along the slit $(-a,a)$, we have

$$\sigma_y-i\tau_{xy}=-\frac{4Kr^2}{\varkappa}J_*(x)X_0^+(x)-\frac{iP}{2\pi}\frac{\varkappa+1}{x}X_0^+(x)$$

Hence, for the case of one inclusion $(\bar a_1=d-ih)$, formulas for the stress components result

$$\sigma_y=-\frac{4K}{\sqrt{\varkappa}}\left(\frac{r}{a}\right)^2\frac{e^{\theta(\varphi_3-\varphi_1)}}{\sqrt{\rho_1\rho_2}}\frac{\rho_3\cos\alpha_1-\rho_1\rho_2((\gamma-q)^2+\beta_1^2)^{-1/2}\cos\alpha_2}{\sqrt{(1-q^2)((\gamma-q)^2+\beta_1^2)}}-\frac{1+\varkappa}{2\pi\sqrt{\varkappa}}\frac{P}{a}\frac{\cos[Q\ln((1+q)/(1-q))]}{\sqrt{1-q^2}}$$

$$\tau_{xy}=-\frac{4K}{\varkappa}\left(\frac{r}{a}\right)^2\frac{e^{\theta(\varphi_3-\varphi_1)}}{\sqrt{\rho_1\rho_2}}\frac{\rho_3\sin\alpha_1-\rho_1\rho_2((\gamma_1-q)^2+\beta_1^2)^{-1/2}\sin\alpha_2}{\sqrt{(1-q^2)((\gamma_1-q)^2+\beta_1^2)}}+\frac{1+\varkappa}{2\pi\sqrt{\varkappa}}\frac{P}{a}\frac{\sin[Q\ln(1+q)/(1-q))]}{\sqrt{1-q^2}}$$

Here

$$h/a=\beta_1,\quad d/a=\gamma_1,\quad x/a=q,\quad \operatorname{tg}\varphi_1=\beta/(\gamma_1-1),\quad \operatorname{tg}\varphi_2=\beta/(\gamma_1+1)$$

$$\operatorname{tg}\varphi_3=(2Q+\beta)/\gamma_1,\quad \operatorname{tg}\theta=\beta/(\gamma_1-q)\quad \rho_1=\sqrt{(\gamma_1-1)^2+\beta^2}$$

$$\rho_2=\sqrt{(\gamma_1+1)^2+\beta_1^2},\quad \rho_3=\sqrt{(2Q+\beta_1)^2+\gamma_1^2},\quad Q=\ln\varkappa/2\pi$$

$$\alpha_1=\frac{1}{2}(\varphi_1+\varphi_2)+Q\ln\left(\frac{\rho_2}{\rho_1}\right)+\theta-\varphi_3-Q\ln\frac{1+q}{1-q}+\frac{\pi}{2}$$

$$\alpha_2=Q\ln\frac{\rho_2}{\rho_1}-Q\ln\frac{1+q}{1-q}+2\theta-\frac{1}{2}(\varphi_1+\varphi_2)+\frac{\pi}{2}$$

N o t e. It is found in deriving the formulas for σ_y and τ_{xy} from the condition $zX_0(z)\to 1$ as $z\to\infty$ that in the function

$$X_0(z)=(z+a)^{-\gamma}(z-a)^{\gamma-1}=\frac{1}{z-a}e^{\gamma\ln u}\quad\left(u=\frac{z-a}{z+a}\right)$$

the argument $\ln u$ equals $\arg u$.

The pressure and shear stresses under a stamp have been computed for $P / 2\pi a = 4 (K / \sqrt{\varkappa}) (r / a)^2$ for the cases $(\beta_1 = 3, \gamma_1 = 0)$, $(\beta_1 = 3, \gamma_1 = 3)$ and the diagrams

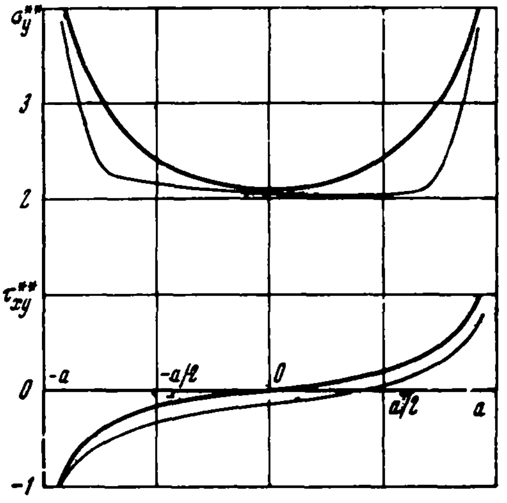

Fig. 3

$\sigma_y^{**} = {}^1/_8 \sqrt{\varkappa}\, K^{-1} (h / r)^2 \sigma_y$ and $\tau_{xy}^{**} = {}^1/_8 \sqrt{\varkappa} K^{-1} (h / r)^2 \tau_y$ are shown in Fig. 3 by fine and heavy lines, respectively.

BIBLIOGRAPHY

1. Muskhelishvili, N. I. : Some Fundamental Problems of the Mathematical Theory of Elasticity. Moscow, Akad. Nauk SSSR Press, 1966.

2. Sherman, D. Y., On the problem of plane strain in nonhomogeneous media. (Proceedings of an I. U. T. A. M. Symposium held in Warsaw, 1958). Pergamon Press, London-N. Y.-Paris-Los Angeles.

3. Galin, L. A., Contact Problems of Elasticity Theory. Moscow, Gostekhizdat, 1953.

Translated by M. D. F.